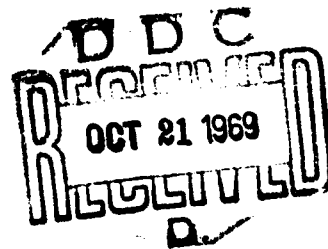# UNITED STATES
NAVAL POSTGRADUATE SCHOOL

ON MODELING A PLANNING, PROGRAMMING

AND BUDGETING SYSTEM

by

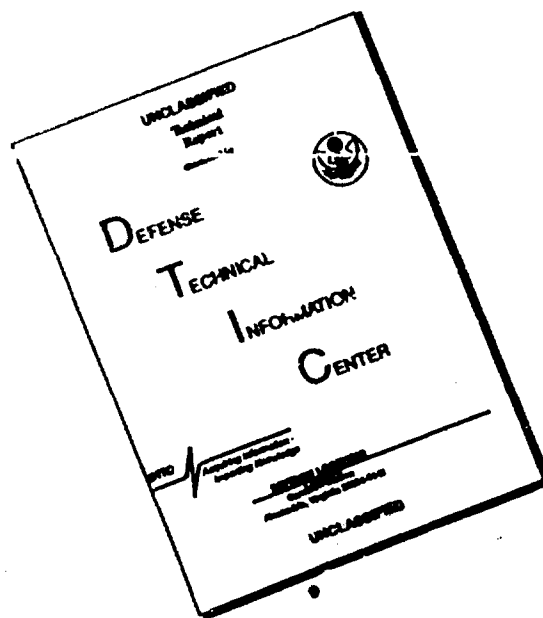
Carl R. Jones

15 July 1969

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ABSTRACT:

A model of the planning and programming aspects of a PPB system is formulated. It incorporates the component parts of a PPB system structure and the multiplicity of benefit and cost measures, both commensurable and incommensurable, so associated. Decision rules characterizing the choice of an efficient program package are derived. These rules indicate the usefulness of a program package numeraire. The comparative statics technique is applied and efficiency substitution and benefit effects derived. A program package cost-benefit function is developed. For a given program package decision, the expansion-contraction paths are also derived and characterized.

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SECTION 1

INTRODUCTION

The notions of program budgeting and the so-called planning, programming, and budgeting system have been discussed for many years. Most of these discussions may be classified as prescriptive in nature. The method of analysis has for the most part been verbal. For these readers unfamiliar with this literature, David Novick's book [1] is an excellent example. So as to avoid confusion with the traditional government budgets and the associated budgeting system, the now-instituted planning, programming, and budgeting system will be referred to here as a planning and programming system.

A review of the literature on program budgeting and planning and programming systems reveals much discussion of its taxonomic structure and advantages over an input oriented approach. When discussing the choice problem of a governmental department head, some attention is usually given to decision rules. These rules equate marginal benefit to marginal cost, which is not a complete surprise, but usually in the context of a single measure of benefit and a single measure of cost. It appears to this author that the more common circumstance is multiple measures of benefit and cost. The former multiplicity is not surprising, while the latter is, at least to the economist. However, the common practice currently is and seemingly will continue to be the use of alternative cost measures.

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When there is a multiplicity of benefit and cost measures, a criterion problem arises. The various authors of the literature usually evoke at least an efficiency criterion; that is, they suggest or rather prescribe that government choice in the planning area be efficient. This criterion is usually evoked and discussed in general terms. This author has not found a single reference which attempts to discuss the resulting decision process structure when an efficiency criterion is evoked in a planning and programming system. It is the purpose of this paper to explore the structure of the decision process associated with the efficiency criterion by formulating a mathematical model of a planning and programming system.

There are two broad approaches, either of which one might adopt in formulating such a model. First it is necessary to identify a planning and programming system as a conflict system. The analysis and lexicon are based on J. March [2]. Professor March considers a conflict system as characterized by the two attributes:

    1. There are consistent basic units.

    2. There is conflict.

In a planning and programming system the consistent basic units are the plethora of study teams and decisionmakers. Some of these units are hierarchically organized and others are of equal rank in the organization. The public press discussions are indicative of the inherent conflict in the system. That is, the preference orderings of the elementary units are mutually inconsistent relative to the resources of the system. It is also clear that the elementary units

can themselves be conflict systems and that the planning and programming system is itself a subsystem of a "larger" conflict system.

Given that a planning and programming system is a conflict system, it is necessary to consider the nature of conflict resolution. Professor March notes that the theories of conflict resolution may be categorized as the imputation of a superordinate goal or as a description of a conflict resolution process. This latter approach, which is analogous to the microeconomic approach to economic systems, would presumably be based on a generalized notion of exchange. In this paper the author has chosen not to approach the formulation of a planning and programming system model using the conflict resolution technique. While research has begun using this approach, it seemed of interest to consider the decision processes using only the imposed superordinate goal of efficiency. That is, in this paper the planning and programming system is modeled based on a superordinate goal of efficiency that is imposed by the department's top management. In the case of interest here, it is not necessary to consider the system operating "as if" there is a superordinate goal; the department's management does require the system to so behave. Using a somewhat different point of view, in this paper the model is formulated abstracting from the internal conflict resolution process. Hence the elementary units are considered not to "suboptimize" using the lexicon of operations research.
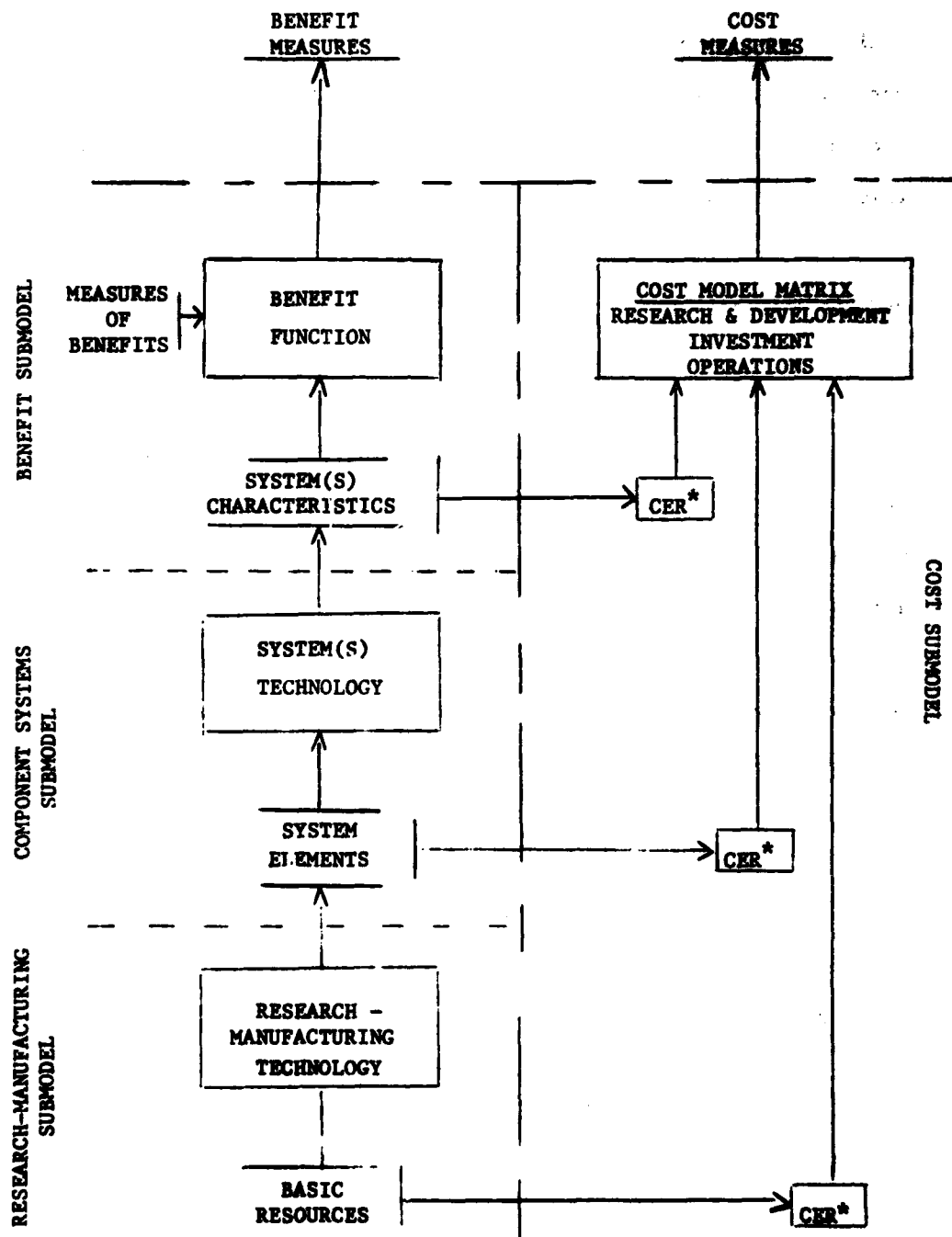
In summary, then, in this paper a model is formulated and exercised which describes the logical structure of the decision processes of a planning and programming system when that system

operates under an efficiency criterion imposed by top management. In Section II the general nature of the problem is discussed. Section III is devoted to the various submodels. The efficiency problem of a department is considered in Section IV. In Section V an overall cost-benefit function for the department is derived. Section VI is devoted to parameter variation results, which is sometimes called comparative statics. The expansion and contraction paths of the department are considered in Section VII. Lastly, the paper is summarized, and some suggestions for future research are given.

SECTION II

THE GENERAL NATURE OF THE PROBLEM

During the planning and programming activities of a governmental department, the department's management must structure its thoughts from its current position through positions in the intermediate years to a position at the end of the planning horizon. As a result, management must give consideration to research and development, manufacturing, and operating aspects of benefit production and the associated costs. Within the context of studying a planning and programming system, Figure 1 is a schematic of the generation of costs and benefits as an interrelated flow among benefit, component system, research and development-manufacturing, and cost submodels. The details of the operational definitions of the variables will be given in the latter part of this section. As can be seen by studying the schematic, basic resources (e.g., engineering hours, raw materials,

BENEFIT
MEASURES

COST
MEASURES

BENEFIT SUBMODEL

MEASURES
OF
BENEFITS →

BENEFIT
FUNCTION

COST MODEL MATRIX
RESEARCH & DEVELOPMENT
INVESTMENT
OPERATIONS

SYSTEM(S)
CHARACTERISTICS

CER*

COMPONENT SYSTEMS
SUBMODEL

SYSTEM(S)
TECHNOLOGY

SYSTEM
ELEMENTS

CER*

COST SUBMODEL

RESEARCH-MANUFACTURING
SUBMODEL

RESEARCH –
MANUFACTURING
TECHNOLOGY

BASIC
RESOURCES

CER*

*Cost Estimating Relationship

SCHEMATIC OF AN IDEALIZED PLANNING AND PROGRAMMING SYSTEM

FIGURE 1

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tooling) are transformed into system elements (e.g., in the military context, tanks, planes, trained personnel). These system elements are the inputs to the component systems submodel. The outputs of this submodel are the system characteristics (e.g., in transportation, range, payload, speed, fuel consumption). These characteristics are produced from the system elements. Finally, characteristics are transformed into values of the system benefit measures (e.g., in poverty programs, expected income distributions).

The inputs to the cost model are characteristics, elements, and resources. By use of cost estimating relationships, the cost model matrix can be computed and the cost measure(s) obtained. All these input types are considered to allow for such phenomenon as learning curves, quantity discounts, and rather detailed disaggregated estimation procedures. The cost model matrix has columns for the time periods of the analysis and rows for the system elements. The columns can be grouped by research and development costs, investment costs, and operating costs, if this is desirable. Some elements of the matrix may, of course, be zero. The cost measure values are computed by pre- and post-multiplication of the matrix by appropriate vectors. For example, if present costs are to be computed, then the premultiplication is by a quantity vector and the post-multiplication is by a vector of discount factors.

An alternative view of a planning and programming system is given by considering it as a taxonomic structure. This is the more common view of program elements fitting into programs which in turn fit into program packages. This taxonomic structure is related to

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the above model by identifying the system elements as the program elements in the planning and programming system taxonomic structure. Those elements may then be grouped into sets as the user requires.

The details of the mathematical programming model associated with the above schematic will be considered in the next section. Before proceeding to that discussion, the variables, the department management's choice objects, will be given operational definitions.

The variables are the benefit measures, the cost measures, the system characteristics, the system elements, and the basic resources. These variables are assumed to have physical-social, time, space, and state-of-nature attributes. In addition to these variables, any exogenous variables introduced in later sections are also assumed to have these attributes. The attributes will be discussed in turn.

The physical attributes of a measure have been discussed before [3]. It is stressed, though, that the same physical and/or social phenomenon can be measured in multiple ways — and they can all be important. For example, Miller, et al., [4] have listed the physical-social (this author's terminology) measures of poverty as income (threshold, relative, share of national income), assets (housing, consumer durables, savings, insurance), and services (education, health, neighborhood amenities, protection, social services, transportation). In considering this model, the reader is urged to regard some of the multiple measures as being associated with the same physical/social phenomenon.

The second attribute is time dating. With this attribute, the same physical/social measure at two different dates will be treated as two different measures. In this fashion, choice object time streams can be associated with a project. It is noted that the time attribute is associated with such measures as present cost and present benefits, since while they are calculated with many dates, they are calculated as of some particular date. Also, the use of a time attribute requires a careful interpretation of capital goods in the model. For example, a system element when conceived of as a physical entity existing over many time periods is included in the model as a sequence of one period stock dimensioned variables. This sequence is constructed such that all of the good in period t is used in producing the same capital good in period $t + 1$. Thus, the interpretation of basic resources in the schematic must include the concept of outputs of one period being inputs in the next.

The third attribute locates the measure of the phenomenon in physical space. Hence, the same physical-social measure at two different locations will be treated as two different measures. A location is determined by categorizing the spatial extension of the phenomenon into elementary regions.

The risk or state-of-nature attribute will be modeled in the Debreusian manner [5]. That is, the future will be modeled as a time sequence of events. At any one date, the events are assertions concerning all that can conceivably happen including natural phenomenon, technological change, political acts, and the like. It is usual to model this as an event tree [6]. While events imply a dating, time will be explicitly discussed for convenience.

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In cost-benefit analysis, particularly as used in the defense department, the scenario has been an important tool. A scenario seems to have no concise definition. However, it is used to mean the background aspects of a given situation. Here, scenario will be used to denote a unicursal path (a path with no steps retraced) through the event tree. It is clear, in a model with only two dates (present and future), scenario and state-of-nature are synonymous. In summary, then, choice variables are defined to have an attribute for the event that could prevail at a given date.

The above concept of state-of-nature is extended here to include the empirical relevance of alternative methods and models. As most practitioners have undoubtedly noticed, discussion concerning the empirical relevance -- "realism" -- of alternative methods and models is often heated and lengthy. It is clear that such disagreement could be resolved by appropriate experimentation and application of scientific procedures. However, since the time frame of the decision does not always allow such experimentation and since the resources for such experimentation may not be available, an attribute of empirical relevance is included in the concept of state-of-nature.

The choice objects, defined with physical-social, time, space, and risk attributes, must also be scaled and given mathematical structure. Here, the details of the scaling will not be considered, but the reader is referred to reference [7]. Rather, each measure is assumed to have an associated ratio scale. This scale is represented by the real numbers.

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SECTION III

THE SUBMODELS

In this section of the paper, the submodels discussed in the previous section will be given a mathematical formulation. The research and development-manufacturing submodel will be considered first. Then the component system, benefit and cost models will be discussed in turn.

THE RESEARCH AND DEVELOPMENT-MANUFACTURING SUBMODEL

As shown in a schematic fashion in Figure 1, the inputs to this submodel are the basic resources and the outputs are the system elements. The basic resources will be designated by the letter x_k ($k = 1, \ldots, K$), the system elements by y_j ($j = 1, \ldots, J$). The technological transformation that represents the R&D-manufacturing process is assumed to be an implicit function involving the elements and resources. This implicit production function is written

$$G(\underline{y}, \underline{x}) = 0.$$

The bar beneath a variable designates a vector.

Various measures of technological trade-off are possible. Of interest here is (1) the trade-off between submodel outputs, (2) the trade-off between submodel inputs, and (3) the effect of an input on an output in the submodel. These trade-offs are shown in Table 1.

NAME OF TRADE-OFF	SYMBOL	FORMULA
RATE OF SYSTEM ELEMENT TRANSFORMATION	$RSETy_j y_\alpha$ $\alpha \neq j$	$\dfrac{\frac{\partial G}{\partial y_\alpha}}{\frac{\partial G}{\partial y_j}} = -\dfrac{\partial y_j}{\partial y_\alpha} = RSETy_j y_\alpha$
RATE OF BASIC RESOURCE SUBSTITUTION	$RBRSx_k x_\alpha$ $\alpha \neq k$	$\dfrac{\frac{\partial G}{\partial x_\alpha}}{\frac{\partial G}{\partial x_k}} = -\dfrac{\partial x_k}{\partial x_\alpha} = RBRSx_k x_\alpha$
MARGINAL PRODUCTIVITY OF RESOURCE k IN THE PRODUCTION OF ELEMENT j	$MPy_j x_k$	$-\dfrac{\frac{\partial G}{\partial x_k}}{\frac{\partial G}{\partial y_j}} = \dfrac{\partial y_j}{\partial x_k} = MPy_j x_k$

Technological Trade-Offs

Research and Development-Manufacturing Submodel

TABLE 1

The subscript α denotes either another output or input than the one subscripted by the j or k , respectively. This α notation will be used throughout the paper

THE COMPONENT SYSTEM SUBMODEL

As shown in a schematic manner in Figure 1, the inputs to this submodel are the system elements and the outputs are the system characteristics. The elements are deisgnated as already discussed, while the letter z_i (i = 1, ..., I) will denote the ith character- istic. The technology embodied in the component systems is represented

by the implicit production function

$$\tau(\underline{z},\underline{y}) = 0 .$$

Table 2 charts the nature of the technological trade-offs applicable to the weapon system technology.

NAME OF TRADE-OFF	SYMBOL	FORMULA
RATE OF SYSTEM CHARACTERIZATION TRANSFORMATION	$RSCTz_i z_\alpha$ $\alpha \neq i$	$\dfrac{\dfrac{\partial F}{\partial z_\alpha}}{\dfrac{\partial F}{\partial z_i}} = -\dfrac{\partial z_i}{\partial z_\alpha} = RSCTz_i z_\alpha$
RATE OF SYSTEM ELEMENT SUBSTITUTION	$RSESy_j y_\alpha$ $\alpha \neq i$	$\dfrac{\dfrac{\partial F}{\partial y_\alpha}}{\dfrac{\partial F}{\partial y_j}} = -\dfrac{\partial y_j}{\partial y_\alpha} = RSESy_j y_\alpha$
MARGINAL COMPONENT SYSTEM PRODUCTIVITY OF ELEMENT j IN THE PRODUCTION OF CHARACTERISTIC i	$MCSPz_i y_j$	$-\dfrac{\dfrac{\partial F}{\partial y_j}}{\dfrac{\partial F}{\partial z_i}} = \dfrac{\partial z_i}{\partial y_j} = MCSPz_i y_j$

Technological Trade-Offs

The Component System Submodel

TABLE 2

THE BENEFIT SUBMODEL

As shown in a schematic manner in Figure 1, the inputs to this submodel are the system characteristics and the outputs the various measures of benefits. The characteristics are denoted as discussed

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and the various benefit measures by E_ℓ ($\ell = 1, \ldots, L$). The technological relationships of the effectiveness submodel are represented by the implicit function

$$H(\underline{E}, \underline{z}) = 0$$

Table 3 contains the information on the trade-offs applicable to the effectiveness submodel.

NAME OF TRADE-OFF	SYMBOL	FORMULA
RATE OF BENEFIT TRANSFORMATION	$RBTE_\ell E_\alpha$	$\dfrac{\frac{\partial H}{\partial E_\alpha}}{\frac{\partial H}{\partial E_\ell}} = - \dfrac{\partial E_\ell}{\partial E_\alpha} = RBTE_\ell E_\alpha$
RATE OF SYSTEM CHARACTERISTIC TRANSFORMATION	$RSCSz_1 z_\alpha$	$\dfrac{\frac{\partial H}{\partial z_\alpha}}{\frac{\partial H}{\partial z_1}} = - \dfrac{\partial z_1}{\partial z_\alpha} = RSCSz_1 z_\alpha$
MARGINAL BENEFIT OF THE ith CHARACTERISTIC IN THE PRODUCTION OF THE ℓth BENEFIT MEASURE	$MBE_\ell z_1$	$-\dfrac{\frac{\partial H}{\partial z_1}}{\frac{\partial H}{\partial E_\ell}} = \dfrac{\partial E_\ell}{\partial z_1} = MBE_\ell z_1$

Technological Trade-Offs

The Benefit Submodel

TABLE 3

THE COST SUBMODEL

As shown schematically in Figure 1, the inputs to this submodel are basic resources $(x_k, k = 1, \ldots, K)$, system elements $(y_j,$

$j = 1, \ldots, J$), system characteristics $(z_i, i = 1, \ldots, I)$ and cost estimating parameters. The outputs are various cost measures. The measures are denoted by $C_m(m = 1, \ldots, M)$ and the cost estimating parameters by r_{mh} $(m = 1, \ldots, M; h = 1, \ldots, H)$. That is, r_{mh} is a cost estimating parameter and, in turn, is related to the statistical parameters in the individual cost estimating equations. The relationship between these variables is expressed as

$$C_m = C_m(\underline{z}, \underline{y}, \underline{x}, \underline{r}_m) .$$

Table 4 contains the interpretation of the various partial slopes of these cost measure functions.

Though the discussion in the remaining sections of this paper will be restricted to consideration of the above cost measure function, some details will now be given to give the reader a better understanding of the functions. As discussed in the previous section, the basic cost model for any measure type is a matrix with columns for time periods and rows for system elements. Since elements have an attribute of time, the cost model matrix, $\underline{\underline{C}}$, has nonzero elements only for the appropriate rows and columns. That is, $\underline{\underline{C}}$ can be partitioned into a diagonal matrix whose nonzero vectors are all on the diagonal. This is sketched in Figure 2. Some of the usual cost measures may be computed for this as shown below.

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NAME OF PARTIAL SLOPE	SYMBOL	FORMULA
TOTAL MARGINAL m^{th} MEASURE COST OF THE ith SYSTEM CHARACTERISTIC	$TMC_m z_i$	$\dfrac{C_m}{z_i} = TMC_m z_i$
TOTAL MARGINAL m^{th} MEASURE COST OF THE jth SYSTEM ELEMENT	$TMC_m y_j$	$\dfrac{C_m}{y_j} = TMC_m y_j$
TOTAL MARGINAL m^{th} MEASURE COST OF THE kth BASIC RESOURCE	$TMC_m x_k$	$\dfrac{C_m}{x_k} = TMC_m x_k$
MARGINAL m^{th} MEASURE COST DUE TO COST ESTIMATING PARAMETER r_{mh}	$MC_m r_{mh}$	$\dfrac{C_m}{r_{mh}} = MCC_m r_{mh}$

Cost Measure Partial Slopes

TABLE 4

$$
\begin{bmatrix}
\underline{c}^1 & \underline{0} & \underline{0} & . & . & . & \underline{0} \\
\underline{0} & \underline{c}^2 & \underline{0} & . & . & . & \underline{0} \\
\underline{0} & \underline{0} & \underline{c}^3 & . & . & . & \underline{0} \\
. & . & . & & & & . \\
. & . & . & & & & . \\
. & . & . & & & & . \\
\underline{0} & \underline{0} & \underline{0} & . & . & . & c^T
\end{bmatrix}
= \underline{\underline{c}}
$$

The Cost Model Matrix
in Partitioned Form

FIGURE 2

where c^t = vector of cost estimating relationships applicable to period t .

Present Cost (PC)

The present cost measure is computed by multiplying each unit cost estimating relationship C_j by its corresponding element y_j ($j \in J^t$ where J^t denotes the indices for time period t). The formula for this is

$$\sum_{j \in J^t} y_j C_j = \underline{y}^T \underline{C} = \underline{O} .$$

The vector $\underline{O}$, a row vector, denotes the total outlays by time period. To complete the present cost calculation, the vector, $\underline{O}$, is multiplied by the vector of discount factors d_t ($t = 1, \ldots, T$). Thus,

$$PC = \underline{y}^T \underline{C} \underline{d} = \underline{O} \cdot \underline{d} = \sum_{t=1}^{T} \sum_{j \in J^t} d_t y_j C_j$$

Total Outlay (TO)

The computation here is the same as for present cost, except the discount factor vector $\underline{d}$ is now merely the sum vector $\underline{1}$.

θ Year System Cost (θC)

The θ year system cost is similar to the total outlay. The difference is that only selected elements are used. The computation is performed by first multiplying a modified identity matrix by the element vector to get a column vector of selected elements. The identity matrix modification is the removal of the diagonal ones for those elements not selected. By formulae the computations are

$\underline{I}^{(S)} \underline{y}$ = column vector of selected elements

$(\underline{I}^{(S)} \underline{y})^T \underline{C}$ = row vector of yearly costs of selected elements

$(\underline{I}^{(S)} \underline{y})^T \underline{C} \, \underline{1}^{(S)}$ = θ year system costs

The symbol $\underline{1}^{(S)}$ denotes the sum vector with zeros for years not of interest in the θ year system costs.

Time Stream of Total Outlay

The measures of interest here are the outlays of costs in each time period. This will be a vector which is computed as $\underline{y}^T\underline{c}$.

Time Stream of Selected Outlays

This measure is like the preceding except that only selected elements are used. The formula is $(\underline{1}^{(S)}\underline{y})^T\underline{c}$.

Unit Costs of System Elements

This measure is a vector measure of unit costs of each of the system elements. The formula is $\underline{C1}$.

In order to give a better understanding of the nature of the partial slopes of the cost measure functions, the next few paragraphs are concerned, first, with the individual elements of the cost model matrix and, second, the computation of appropriate partial slopes for some of the measures just discussed.

The individual CER is formulated as

$$c_j = f^{jt}(\underline{z},\underline{y},\underline{x},\underline{1}_{jt}).$$

The partial derivatives of this function with respect to the characteristics, elements, and resources measure the respective unit marginal costs. The partial derivative with respect to the cost coefficient γ_{hjt} measures the marginal j^{th} element cost with respect to its own k^{th} cost coefficient. When considering the marginal effect on system cost, the total marginal cost, all such unit marginal effects must be included. When the present cost measure (PC) is used, the computation is for the ith characteristic.

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$$TMC_{PC}z_i = \frac{\partial(\underline{y}^T\underline{C}\underline{d})}{\partial z_i} = \frac{\partial\left(\sum_{t=1}^{T}\sum_{i\epsilon J}t_t d_t y_j f^{jt}(\underline{z},\underline{y},\underline{x},\underline{y}_{jt})\right)}{\partial z_i}$$

$$= \sum_{t=1}^{T}\sum_{j\epsilon J}t_t d_t y_j \frac{\partial f^{jt}}{\partial z_i}$$

The result for total marginal cost is the weighted sum of the unit
marginal costs with the weights being quantity of elements and the
discount factor. When system elements are considered, the formula
is

$$TMC_{PC}y_j = \frac{\partial(\underline{y}^t\underline{C}\underline{d})}{\partial y_j} = \sum_{t=1}^{T}\sum_{j\epsilon J}t_t d_t (f^{jt} + y_j \frac{\partial f^{jt}}{\partial y_j})$$

In this case there are two effects, since elements appear directly
and as part of CER's. This same general pattern of weighted unit
effects occurs for the other measures. This weighting is the reason
for defining the partial slopes of the cost measure functions as
total partial slopes. They are, in turn, usually complicated expres-
sions.

SECTION IV

EFFICIENT DEPARTMENTAL PROGRAMS

Using the submodels discussed in the previous section, the decision
processes of the department can be studied. The planning and pro-
gramming system modeled in this paper operates under a top management
imposed superordinate efficiency criterion. Using this criterion,

alternative solutions are represented as vectors of benefit and cost levels such that there is no vector that will give more of one component without giving less of another. The technique of vector maximization is especially useful for these types of problems and will be used in this paper. Background material on this technique is given in references [8] and [9].

In formal terms, the efficiency problem is

$$\text{"Max"} \quad \left[\frac{\underline{E}}{\underline{C}}\right]$$

s.t.
$$H(\underline{E},\underline{z}) = 0$$
$$F(\underline{z},\underline{y}) = 0$$
$$G(\underline{y},\underline{x}) = 0$$
$$\underline{E},\underline{x},\underline{y},\underline{z} \geq 0$$

As the reader has undoubtedly observed, there are no restrictions on the signs of costs. This could be accomplished by adding additional constraints. This is not done here as no essential notion is lost by its exclusion. The usual assumptions concerning differentiability, constraint qualifications, and concavity/convexity are assumed. The Lagrangian of this problem is

$$\mathcal{L}(\underline{E},\underline{x},\underline{y},\underline{z},\lambda) = \sum_{\ell=1}^{L} \phi_\ell E_\ell + \sum_{m=1}^{M} \psi_m C_m + \lambda_1 F(\underline{z},\underline{y}) +$$

$$\lambda_2 G(\underline{y},\underline{x}) + \lambda_3 H(\underline{E},\underline{z}) .$$

The necessary conditions for a maximum are as follows. A variable of the maximization problem, which appears below as a sub-script, denotes a partial derivative with respect to that variable.

(1) $\phi_\ell + \lambda_3 H_{E_\ell} \leq 0 \qquad \ell = 1, \ldots, L$

$\hat{E}_\ell (\phi_\ell + \lambda_3 H_{E_\ell}) = 0 \qquad \hat{E}_\ell \geq 0$

(2) $\sum_{m=1}^{M} \psi_m C_{mz_i} + \lambda_1 F_{z_i} + \lambda_3 H_{z_i} \leq 0 \qquad i = 1, \ldots, I$

$\hat{z}_i \left(\sum_{m=1}^{M} \psi_m C_{mz_i} + \lambda_1 F_{z_i} + \lambda_3 H_{z_i} \right) = 0 \qquad \hat{z}_i \geq 0$

(3) $\sum_{m=1}^{M} \psi_m C_{my_j} + \lambda_1 F_{y_j} + \lambda_2 G_{y_j} \leq 0 \qquad j = 1, \ldots, J$

$\hat{y}_j \left(\sum_{m=1}^{M} \psi_m C_{my_j} + \lambda_1 F_{y_j} + \lambda_2 G_{y_j} \right) = 0 \qquad \hat{y}_j \geq 0$

(4) $\Sigma \psi_m C_{mx_k} + \lambda_2 G_{x_k} \leq 0 \qquad k = 1, \ldots, K$

$\hat{x}_k (\Sigma \psi_m C_{mx_k} + \lambda_2 G_{x_k}) = 0 \qquad \hat{x}_k \geq 0$

(5) $F(\underline{z}, \underline{y}) = 0$

(6) $G(\underline{y}, \underline{x}) = 0$

(7) $H(\underline{E}, \underline{z}) = 0$

It is noted that the number of dependent variables $(\underline{E}, \underline{x}, \underline{y}, \underline{z}, \underline{\lambda})$ equals the number of equations.

The topic to be discussed next is the decision rules which can be derived from the necessary conditions. These decision rules are necessary for a maximum, and they are sufficient if the full concavity (convexity) assumptions are made. The rules presented below will be for the case where all variables are at a positive level and all the necessary conditions are equations. Further, the equations can be manipulated in various ways and only one possibility is presented here.

Decision Rule 1

Using equations 1, it is found that

$$\frac{\phi_\alpha}{\phi_\ell} = \frac{-\lambda_3 H_{E_\alpha}}{-\lambda_3 H_{E_\ell}} \Rightarrow \frac{\phi_\alpha}{\phi_\ell} = \frac{H_{E_\alpha}}{H_{E_\ell}} = RBTE_\ell E_\alpha.$$

This rule means that at an optimum the rate of transformation of benefit measures is equal to the appropriate ratio of ϕ's. In the economic literature the ϕ's are known as efficiency prices and they shall be so interpreted here. Since only relative efficiency prices are of interest as these measure the rates of benefit transformation, benefit E_ℓ is selected as numéraire.

Decision Rule 2

Using equations 1, 2, and the choice of numéraire, it is found that

$$\frac{\sum_{m=1}^{M} \left(\frac{\psi_m}{\phi_\ell}\right)(TMC_m z_\alpha) + (MBE_\ell z_\alpha)}{\sum_{m=1}^{M} \left(\frac{\psi_m}{\phi_\ell}\right)(TMC_m z_i) + (MBE_\ell z_i)} = \frac{F_{z_\alpha}}{F_{z_i}} = (RSCTz_i z_\alpha)$$

This rule is more easily interpreted if the C_m's are constrained to be negative numbers. The ψ's and ϕ's are strictly positive as shown by Karlin [Ref. 8, p. 217]. Then the first term in the numerator is the total variation in cost due to a change in the αth characteristic. The sum of these terms, then, is the net change in units of benefit units of ℓ due to a direct effect on the benefit $(MBE_\ell z_\alpha)$ and an indirect effect due to the cost measures. Overall, the rule says that the ratio of net variations in marginal benefit should equal the rate of characteristics transformation.

Decision Rule 3

Using equations 3 and the other decision rules, it is found that

$$\frac{\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{my_\alpha} + \left(\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{mz_i} + (MBE_\ell z_i)\right)(MCSPz_i y_\alpha)}{\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{my_j} + \left(\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{mz_i} + (MBE_\ell z_i)\right)(MCSPz_i y_j)} =$$

$$\frac{G_{y_\alpha}}{G_{y_j}} = RSETy_j y_\alpha$$

The first term in the numerator measures the marginal effect on all costs (in units of E_ℓ) of a change in y_α. The second term first measures the effect of y_α on $z_i (MCSPz_i y_\alpha)$, then the effect of z_i on net units of E_ℓ. Again, there is an indirect effect of y_α on costs (first term) and a direct effect transformed to net units of E_ℓ. The overall numerator can be thought of as the net efficiency value of an additional unit of y_α measured in units of E_ℓ. The ratio of the net efficiency value of additional units of y_α and y_j must equal the rate of transformation of y_j and y_α at an optimum.

Decision Rule 4

Using equations 4, the decision rule is

$$\frac{\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{mx_\alpha}}{\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{mx_k}} = \frac{G_{x_\alpha}}{G_{x_k}} = MBRSx_k x_\alpha$$

The right side of this decision rule is the ratio of two total marginal costs. So the rule says to equate the rate of substitution to the

ratio of total marginal costs. The costs are again measured in units
of E_ℓ .

As can be seen in this sample of decision rules, the application
of an efficiency criteria to the outputs of a planning and programming
system leads to rather complicated decision rules. The framework for
considering such problems includes the use of efficiency prices and a
numéraire. Some reduction in complication would occur if the unit of
account were consciously chosen and used in interpreting the decision
rules.

SECTION V

THE IMPLICIT COST-BENEFIT FUNCTION

In many discussions of planning and programming systems and the
associated departmental management problem some "nice" function of
cost and benefits is assumed. For example, the implicit function

$$H(\underline{E},\underline{C}) = 0$$

is sometime used in the discussion. It is the purpose of this part
of the paper to discuss the relationship of such an implicit form to
the previous model.

The necessary conditions for the efficiency (vector, maximum)
problem (equations IV.1 - IV.7) can be solved for the choice variables
as functions of the efficiency prices and the cost coefficients. That
is, by use of the implicit function theorem applied to equations (1)
through (7), the following equations can be developed.

$$\underline{x} = \underline{X}(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\underline{y} = \underline{Y}(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\underline{z} = \underline{Z}(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\underline{E} = \underline{E}(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\lambda_1 = \lambda_1(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\lambda_2 = \lambda_2(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\lambda_3 = \lambda_3(\underline{\phi}, \underline{\psi}, \underline{r})$$

In addition, it is known that

$$\underline{C} = \underline{C}(\underline{x}, \underline{y}, \underline{z}, \underline{r})$$

Substituting for $\underline{x}$, $\underline{y}$, $\underline{z}$, this equation becomes

$$\underline{C} = \underline{C}(\underline{X}(\underline{\phi}, \underline{\psi}, \underline{r}), \underline{Y}(\underline{\phi}, \underline{\psi}, \underline{r}), \underline{Z}(\underline{\phi}, \underline{\psi}, \underline{r}), \underline{r})$$

This set of equations, together with

$$\underline{E} = \underline{E}(\underline{\phi}, \underline{\psi}, \underline{r})$$

parametrically determine the cost-benefit surface.

These equations for $\underline{C}$ and $\underline{E}$ as functions of $\underline{\phi}$, $\underline{\psi}$ parametrically determine the cost-benefit surface (10, p. 371-375) since the sum of the ϕ's and ψ's is one. This latter theorem for vector maximum problems is proved by Karlin (8, p. 216-218). Hence, again using the implicit function theorem, $L + M - 1$ of the ϕ's and ψ's can be solved for as functions of the $L + M - 1$ C's and E's. In turn, these may be substituted into the remaining equation yielding, for example, the explicit form

$$E_L = f(E_1, \ldots, E_{L-1}, \underline{C})$$

which is easily transformed into implicit form. Hence, the vector maximum formulation permits the development of an implicit cost-benefit function.

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Now that the cost-benefit surface is known in implicit function form, at least locally, it is of interest to study the qualitative properties of the surface. That is, the signs of output transformations, input substitutions, and marginal productivities are of interest. This line of research has not yet been pursued. It is noted that it involves a repeated application of the corollary to the implicit function theorem on the slopes of implicit functions (11).

SECTION VI

EFFICIENT SOLUTIONS AND PARAMETER VARIATIONS

The effect on the efficient solution values of the variables in the model of variation in the parameters of the model is now considered. This sensitivity analysis is performed in the usual manner by considering the first order conditions discussed in Section IV as implicitly defining a relationship between the variables and the parameters. Inequalities among the first order conditions present no difficulties as shown by King [12]. The slopes of the implicit variable-parameter functions are the point of investigation. The results of the investigation will be to show that the overall variation in the x's, y's, and z's can be considered as an efficiency substitution effect and a benefit effect. The first parameter of interest will be a cost coefficient r. The next parameter will be an efficiency price.

Differentiating the equality necessary conditions yields the equations

$$
\begin{bmatrix}
A_{11}^{L\times L} & A_{12}^{L\times I} & 0^{L\times J} & 0^{L\times K} & 0^{L\times 1} & 0^{L\times 1} & A_{17}^{L\times 1} \\
A_{21}^{I\times L} & A_{22}^{I\times I} & A_{23}^{I\times J} & A_{24}^{I\times K} & A_{25}^{I\times 1} & A^{I\times 1} & 0_{27}^{I\times 1} \\
0^{J\times L} & A_{32}^{J\times I} & A_{33}^{J\times J} & A_{34}^{J\times K} & A_{35}^{J\times 1} & A_{36}^{J\times 1} & 0^{J\times 1} \\
0^{K\times L} & A_{42}^{K\times I} & A_{43}^{K\times J} & A_{44}^{K\times K} & 0^{K\times 1} & A_{46}^{K\times 1} & 0^{K\times 1} \\
0^{1\times L} & A_{25}^{1\times I} & A_{35}^{1\times J} & 0^{1\times K} & 0 & 0 & 0 \\
0^{1\times L} & 0^{1\times L} & A_{36}^{1\times J} & A_{46}^{1\times K} & 0 & 0 & 0 \\
A_{17}^{1\times L} & A_{27}^{1\times I} & 0^{1\times J} & 0^{1\times K} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial E}{\partial r} \\
\frac{\partial z}{\partial r} \\
\frac{\partial y}{\partial r} \\
\frac{\partial x}{\partial r} \\
\frac{\partial \lambda_1}{\partial r} \\
\frac{\partial \lambda_2}{\partial r} \\
\frac{\partial \lambda_3}{\partial r}
\end{bmatrix}
=
\begin{bmatrix}
0^{L\times 1} \\
b_2^{I\times 1} \\
b_3^{J\times 1} \\
b_4^{K\times 1} \\
0 \\
0 \\
0
\end{bmatrix}
\qquad (I)
$$

The details of these equations can be seen in Table 5. For convenience of expression, set I is written as

$$\underline{\underline{A}}\,\underline{q} = \underline{b}.$$

The solution to this set of equations is then, in formal terms,

$$\underline{q} = \underline{\underline{A}}^{-1}\underline{b}\,.$$

To understand this solution, it is necessary to consider the following efficiency problem. In this problem the idea is to vectorially maximize costs subject to a fixed level of benefits and the various technological transformations. In formal terms,

$$\text{"Max"}\ \sum_{m=1}^{M} \psi_m C_m$$

$$\text{s.t.}$$
$$H(\dot{E},z) = 0$$
$$F(z,y) = 0$$
$$G(y,x) = 0$$
$$x,y,z \geq 0$$

where $\dot{E}$ designates the fixed level of all effectiveness measures. The notion of maximum is used as costs are treated as negative numbers. The Lagrangian for this problem is

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Equation Set I Details with Typical Elements

	z			y		x	λ_1	λ_2	λ_3		
(1)	$\left(\lambda_3 H_{E_i E_i}, \lambda_3 H_{E_i z_i}\right)$			(0)		(0)	(0)	(0)	$\left(H_{E_i}\right)$		$\dfrac{\partial z_2}{\partial \gamma}$
(2)	$\left(\lambda_3 H_{z_i E_i}, \Sigma \psi_m C_{mz_i z_i} + \lambda_1 F_{z_i z_i} + \lambda_3 H_{z_i z_i}\right)$		$\left(\Sigma \psi_m C_{mz_i y_j} + \lambda_1 F_{z_i y_j}\right)$		$\left(\Sigma \psi_m C_{mz_i x_k}\right)$		$\left(F_{z_i}\right)$	(0)	$\left(H_{z_i}\right)$		$\dfrac{\partial z_1}{\partial \gamma}$
(3)	(0)		$\left(\Sigma \psi_m C_{my_j z_i} + \lambda_1 F_{y_j z_i}\right)$		$\left(\Sigma \psi_m C_{my_j y_j} + \lambda_1 F_{y_j y_j} + \lambda_2 G_{y_j y_j}\right)$	$\left(\Sigma \psi_m C_{my_j x_k} + \lambda_2 G_{y_j x_k}\right)$	(0)	$\left(F_{y_j}\right)$	$\left(G_{y_j}\right)$		$\dfrac{\partial y_1}{\partial \gamma}$
(4)	(0)		$\left(\Sigma \psi_m C_{mx_k z_i}\right)$		$\left(\Sigma \psi_m C_{mx_k y_j} + \lambda_2 G_{x_k y_j}\right)$	$\left(\Sigma \psi_m C_{mx_k x_k} + \lambda_2 G_{x_k x_k}\right)$	(0)	(0)	(0)		$\dfrac{\partial x_k}{\partial \gamma}$
(5)	(0)		$\left(F_{z_i}\right)$		$\left(F_{y_j}\right)$	(0)	(0)	(0)	(0)		$\dfrac{\partial \lambda_1}{\partial \gamma}$
(6)	(0)		(0)		$\left(G_{y_j}\right)$	$\left(G_{x_k}\right)$	(0)	(0)	(0)		$\dfrac{\partial \lambda_2}{\partial \gamma}$
(7)	$\left(H_{E_i}\right)$		$\left(H_{z_i}\right)$		(0)	(0)	(0)	(0)	(0)		$\dfrac{\partial \lambda_3}{\partial \gamma}$

$$= \left((0),\; (-\Sigma \psi_m C_{mz_i\gamma}),\; (-\Sigma \psi_m C_{my_j\gamma}),\; (-\Sigma \psi_m C_{mx_k\gamma}),\; (0),\; (0),\; (0)\right)^T$$

Equation Set I Details with Typical Elements

TABLE 5

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$$\mathcal{L}^*(\underline{x},\underline{y},\underline{z},\underline{\mu}) = \sum_{m=1}^{M} \psi_m C_m + \mu_1 F(\underline{z},\underline{y}) + \mu_2 G(\underline{y},\underline{x}) + \mu_3 H(\overset{\bullet}{\underline{E}},\underline{z}).$$

Again, the usual mathematical assumptions are made. The necessary conditions are

(A) $\quad \sum\limits_{m=1}^{M} \psi_m C_{mz_i} + \mu_1 F_{z_i} + \mu_3 H_{z_i} \leq 0 \qquad i = 1,\ldots,I$

$\quad \hat{z}_i \left(\sum\limits_{m=1}^{M} \psi_m C_{mz_i} + \mu_1 F_{z_i} + \mu_3 H_{z_i} \right) = 0 \qquad \hat{z}_i \geq 0$

(B) $\quad \sum\limits_{m=1}^{M} \psi_m C_{my_j} + \mu_1 F_{y_j} + \mu_2 G_{y_j} \leq 0 \qquad j = 1,\ldots,J$

$\quad \hat{y}_j \left(\sum\limits_{m=1}^{M} \psi_m C_{my_j} + \mu_1 F_{y_j} + \mu_2 G_{y_j} \right) = 0 \qquad \hat{y}_j \geq 0$

(C) $\quad \sum\limits_{m=1}^{M} \psi_m C_{mx_k} + \mu_2 G_{x_k} \leq 0 \qquad k = 1,\ldots,K$

$\quad \hat{x}_k \left(\sum\limits_{m=1}^{M} \psi_m C_{mx_k} + \mu_2 G_{x_k} \right) = 0 \qquad \hat{x}_k \geq 0$

(D) $\quad F(\underline{z},\underline{y}) = 0$

(E) $\quad G(\underline{y},\underline{x}) = 0$

(F) $\quad H(\overset{\bullet}{\underline{E}},\underline{z}) = 0$

The number of variables (x's, y's, z's, μ's) can be shown to equal the number of equations.

The relationships of these conditions to the original maximum problem are first studied by means of the Lagrange multipliers. Using equations (C) and (4), it can be shown that $\hat{\lambda}_2 = \hat{\mu}_2$ if the partial derivatives are evaluated at the same point. Equations (B) and (3) are used in conjunction with $\hat{\lambda}_2 = \hat{\mu}_2$ to obtain the theorem that $\hat{\mu}_1 = \hat{\lambda}_1$. Equations (A) and (2) in conjunction with $\hat{\mu}_1 = \hat{\lambda}_1$ are used

to obtain $\mu_3 = \lambda_3$. Of course, all these equalities assume the partial derivatives are evaluated at the same point.

The second relationship between the efficiency problem and the maximum problem concerns the decision rules. With the relationship of the Lagrange multipliers it is clear that equations (A), (B), and (C) are the same as (2), (3), and (4). Hency, the decision rules are the same if derived only from these equations. The decision rules of the last section directly use (1), but this need not have been the case. It is concluded, then, that where applicable the two formulations led to the same decision rules.

To investigate the efficiency substitution effect, it is necessary to differentiate the necessary conditions with respect to the cost function parameter r. This procedure yields the equations

$$
\begin{bmatrix}
B_{11}^{I \times I} & B_{12}^{I \times J} & B_{13}^{I \times K} & B_{14}^{I \times 1} & 0^{I \times 1} & B_{16}^{I \times 1} \\
B_{21}^{J \times I} & B_{22}^{J \times J} & B_{23}^{J \times K} & B_{24}^{J \times 1} & B_{25}^{J \times 1} & 0^{J \times 1} \\
B_{31}^{K \times I} & B_{32}^{K \times J} & B_{33}^{K \times K} & 0^{K \times 1} & B_{35}^{K \times 1} & 0^{K \times 1} \\
B_{41}^{1 \times I} & B_{42}^{1 \times J} & 0^{1 \times K} & 0 & 0 & 0 \\
0^{1 \times I} & B_{52}^{1 \times J} & B_{53}^{1 \times K} & 0 & 0 & 0 \\
B_{61}^{1 \times I} & 0^{1 \times J} & 0^{1 \times K} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial z}{\partial r} \\[4pt]
\dfrac{\partial y}{\partial r} \\[4pt]
\dfrac{\partial x}{\partial r} \\[4pt]
\dfrac{\partial \mu_1}{\partial r} \\[4pt]
\dfrac{\partial \mu_2}{\partial r} \\[4pt]
\dfrac{\partial \mu_3}{\partial r}
\end{bmatrix}
=
\begin{bmatrix}
b_1^{I \times 1} \\[4pt]
b_2^{J \times 1} \\[4pt]
b_3^{K \times 1} \\[4pt]
0 \\[4pt]
0 \\[4pt]
0
\end{bmatrix}
\qquad \text{(II)}
$$

In more compact notation:

$$
B s = t \Rightarrow \hat{s} = B^{-1} t
$$

The details of this system of equations are shown in Table 6.

Table 6 — rotated, heavily degraded.

	u_1 u_2 u_3	x	y	z
(1) $\dfrac{\partial E}{\partial Y}$	$\left(F_{z_i}\right)(0)\,'(H_{z_i})$	$\left(\Sigma\psi_m C_{mx_i x_k}\right)$	$\left(\Sigma\psi_m C_{mx_i y_j} + u_1 F_{z_i y_j}\right)$	$\left(\Sigma\psi_m C_{mx_i z_i} + u_1 F_{z_i z_i} + u_3 F_{z_i z_i}\right)$
(2) $\dfrac{\partial X}{\partial Y}$	$F_{y_j}\left(G_{y_j}\right)(0)$	$\left(\Sigma\psi_m C_{my_j x_k} + u_2 G_{y_j z_k}\right)$	$\left(\Sigma\psi_m C_{my_j} + u_1 F_{y_j y_j} + u_2 G_{y_j y_j}\right)$	$\left(\Sigma\psi_m C_{my_j z_i} + u_1 F_{y_j z_i}\right)$
(3) $\dfrac{\partial W}{\partial Y}$	$(0)\left(G_{z_k}\right)(0)$	$\left(\Sigma\psi_m C_{mz_k x_k} + u_2 G_{z_k z_k}\right)$	$\left(\Sigma\psi_m C_{mz_k y_j} + u_2 G_{z_k y_j}\right)$	$\left(\Sigma\psi_m C_{mz_k z_i}\right)$
(4) $\dfrac{\partial u_1}{\partial Y}$	$(0)\ (0)\ (0)$	(0)	$\left(F_{y_j}\right)$	$\left(F_{z_i}\right)$
(5) $\dfrac{\partial u_2}{\partial Y}$	$(0)\ (0)\ (0)$	$\left(G_{z_k}\right)$	$\left(G_{y_j}\right)$	(0)
(6) $\dfrac{\partial u_3}{\partial Y}$	$(0)\ (0)\ (0)$	(0)	(0)	$\left(H_{z_i}\right)$

$$\left(\left(-\Sigma\psi_m C_{mx_i}\right);\ \left(-\Sigma\psi_m C_{my_j}\right),\left(-\Sigma\psi_m C_{mz_k}\right),(0),(0),(0)\right)^T$$

Equation Set II Details with Typical Elements

TABLE 6

30

Inspection of I and II shows that II is, in fact, a submatrix of I. Using this information, I is written as the partitioned matrix:

$$
\begin{bmatrix}
A_{11}^{LxL} & A_{12}^{LxI} & \underline{0}^{LxJ} & \underline{0}^{LxK} & \underline{0}^{Lx1} & \underline{0}^{Lx1} & A_{17}^{Lx1} \\
A_{21}^{IxL} & & & & & & \\
\underline{0}^{JxL} & & & & & & \\
\underline{0}^{KxL} & & & \underline{B}^{I+J+K+3xI+J+K+3} & & & \\
\underline{0}^{1xL} & & & & & & \\
\underline{0}^{1xL} & & & & & & \\
A_{71}^{1xL} & & & & & &
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial \hat{R}}{\partial r} \\[4pt]
\dfrac{\partial z}{\partial r} \\[4pt]
\dfrac{\partial y}{\partial r} \\[4pt]
\dfrac{\partial x}{\partial r} \\[4pt]
\dfrac{\partial \lambda_1}{\partial r} \\[4pt]
\dfrac{\partial \lambda_2}{\partial r} \\[4pt]
\dfrac{\partial \lambda_3}{\partial r}
\end{bmatrix}
=
\begin{bmatrix}
\underline{0}^{Lx1} \\[4pt]
\\
\\
\underline{t} \\
\\
\\
\end{bmatrix}
$$

the equations can be written

$$
\begin{bmatrix}
A_{11}^{LxL} & \underline{0}^{LxI+J+K+3} \\
\underline{P}^{I+J+K+3xL} & \underline{B}^{I+J+K+3xI+J+K+3}
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial \hat{R}}{\partial r} \\[4pt]
\zeta
\end{bmatrix}
=
\begin{bmatrix}
\underline{0}^{Lx1} \\[4pt]
\underline{t}
\end{bmatrix}
$$

The solution is:

$$
\begin{bmatrix}
\dfrac{\partial \underline{R}}{\partial r} \\[6pt]
\zeta
\end{bmatrix}
=
\begin{bmatrix}
(\underline{A}_{11}-\underline{Q}\,\underline{B}^{-1}\underline{P})^{-1\ LxL} & -(\underline{A}_{11}-\underline{Q}\,\underline{B}^{-1}\underline{P})^{-1}\underline{Q}\underline{B}^{-1\ \ LxI+J+K+3} \\[6pt]
-\underline{B}^{-1}\underline{P}(\underline{A}_{11}-\underline{Q}\,\underline{B}^{-1}\underline{P})^{-1\ \ I+J+K+3xL} & \underline{B}^{-1}+\underline{B}^{-1}\underline{P}(\underline{A}_{11}-\underline{Q}\,\underline{B}^{-1}\underline{P})^{-1}\underline{Q}\underline{B}^{-1\ \ I+J+K+3xI+J+K+3}
\end{bmatrix}
\begin{bmatrix}
\underline{0}^{Lx1} \\[6pt]
\underline{t}^{I+J+K+3x1}
\end{bmatrix}
$$

so

$$
\begin{bmatrix}
\dfrac{\partial \underline{R}}{\partial r} \\[6pt]
\zeta
\end{bmatrix}
=
\begin{bmatrix}
-(\underline{A}_{11} - \underline{Q}\underline{B}^{-1}\underline{P})^{-1}\underline{Q}\underline{B}^{-1}\underline{t} \\[6pt]
\underline{B}^{-1}\underline{t} + \underline{B}^{-1}\underline{P}(\underline{A}_{11} - \underline{Q}\underline{B}^{-1}\underline{P})^{-1}\underline{Q}\underline{B}^{-1}\underline{t}
\end{bmatrix}
$$

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Notice that the solution for $\frac{\partial E}{\partial r}$ also appears in the solution for ζ. Also $\hat{s}$ appears. Explicitly then, the solution for ζ is:

$$\zeta = \hat{s} - B^{-1} L \frac{\partial E}{\partial r}$$

The first term of this equation is the efficiency substitution effect, and the second term is the benefit effect. Thus, a change in a cost function parameter can be considered to have two additive components. The first component is the variation in the z, x, or y due to the variation in the cost coefficient holding the effectiveness level constant. The second component is the effect on the z, x, or y due to the effect on benefit due to the cost coefficient. This latter component effect is due to variations in the z, x, y in the technologies and cost functions.

In a manner analogous to traditional economic theory, substitutes and complements can be defined. For efficiency substitutes and complements the definitions are:

Efficiency Substitutes

Two characteristics (elements, resources) are called efficiency substitutes for cost function parameter r_{mh} if

$$\left.\frac{\partial z_i}{\partial r_{mh}}\right|_E > 0 \quad \text{and} \quad \left.\frac{\partial z_\alpha}{\partial r_{mh}}\right|_E < 0 \qquad i \neq \alpha$$

Efficiency Complements

Two characteristics (elements, resources) are called efficiency complements for cost function parameter r_{mh} if

$$\left.\frac{\partial z_i}{\partial r_{mh}}\right|_E > 0 \quad \text{and} \quad \left.\frac{\partial z_\alpha}{\partial r_{mh}}\right|_E > 0 \qquad i \neq \alpha$$

32

The next parameter to consider is the one associated with the efficiency prices. This problem is of interest since it is the variation in these prices which "sweeps out" the cost-benefit surface.

Again, the necessary conditions for the vector maximum are differentiated with respect to the variable of interest, which is now the efficiency price ψ_m associated with the mth cost measure. The following equations are obtained.

$$
\Delta
\begin{bmatrix}
\dfrac{\partial E}{\partial \psi_m} \\[2ex]
\dfrac{\partial E}{\partial \psi_m} \\[2ex]
\dfrac{\partial Y}{\partial \psi_m} \\[2ex]
\dfrac{\partial x}{\partial \psi_m} \\[2ex]
\dfrac{\partial \lambda_1}{\partial \psi_m} \\[2ex]
\dfrac{\partial \lambda_2}{\partial \psi_m} \\[2ex]
\dfrac{\partial \lambda_3}{\partial \psi_m}
\end{bmatrix}
=
\begin{bmatrix}
0 \\[2ex]
-C_{mz} \\[2ex]
-C_{my} \\[2ex]
-C_{mx} \\[2ex]
0 \\[2ex]
0 \\[2ex]
0
\end{bmatrix}
$$

For convenience this set of equations is written as

$$\Delta\, \underline{u} = \underline{v}$$

The solution in formal terms is

$$\underline{u} = \Delta^{-1}\underline{v}$$

33

Before continuing with this development, it is noted that the effect of a variation in ψ_m on the values of the cost measures is given by

$$\frac{\partial C_m}{\partial \psi_m} = \sum_{k=1}^{K} \frac{\partial C_m}{\partial x_k} \frac{\partial x_k}{\partial \psi_m} + \sum_{j=1}^{J} \frac{\partial C_m}{\partial y_j} \cdot \frac{\partial y_j}{\partial \psi_m} + \sum_{i=1}^{I} \frac{\partial C_m}{\partial z_i} \cdot \frac{\partial z_i}{\partial \psi_m}$$

$$(m = 1, \ldots, M)$$

Continuing with the main development again, the result of applying the procedure to the efficiency problem is the following set of equations.

$$
\mathbf{A}
\begin{bmatrix}
\dfrac{\partial \mathbf{z}}{\partial \psi_m} \\[1em]
\dfrac{\partial \mathbf{y}}{\partial \psi_m} \\[1em]
\dfrac{\partial \mathbf{x}}{\partial \psi_m} \\[1em]
\dfrac{\partial \mu_1}{\partial \psi_m} \\[1em]
\dfrac{\partial \mu_2}{\partial \psi_m} \\[1em]
\dfrac{\partial \mu_3}{\partial \psi_m}
\end{bmatrix}
=
\begin{bmatrix}
-C_{mz} \\[1em]
-C_{my} \\[1em]
-C_{mx} \\[1em]
0 \\[1em]
0 \\[1em]
0
\end{bmatrix}
$$

In more compact notation:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

with the formal solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

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Again, this equation set is seen to be a subset of the previous, and the original set may be written in partitioned form as

$$
\begin{bmatrix} A_{11} & Q \\ P & R \end{bmatrix}
\begin{bmatrix} \dfrac{\partial E}{\partial \psi_m} \\ \delta \end{bmatrix}
=
\begin{bmatrix} 0 \\ \ell \end{bmatrix}
$$

where

$$
\delta =
\begin{bmatrix}
\dfrac{\partial z}{\partial \psi_m} \\
\dfrac{\partial y}{\partial \psi_m} \\
\dfrac{\partial x}{\partial \psi_m} \\
\dfrac{\partial \lambda_1}{\partial \psi_m} \\
\dfrac{\partial \lambda_2}{\partial \psi_m} \\
\dfrac{\partial \lambda_3}{\partial \psi_m}
\end{bmatrix}
$$

The solution is

$$
\begin{bmatrix} \dfrac{\partial E}{\partial \psi_m} \\ \delta \end{bmatrix}
=
\begin{bmatrix}
-(A_{11} - Q R^{-1} P)^{-1} Q R^{-1} \ell \\
R^{-1} \ell + R^{-1} P (A_{11} - Q R^{-1} P)^{-1} Q R^{-1} \ell
\end{bmatrix}
$$

so the solution for δ may be written as

$$
\delta = x - R^{-1} P \frac{\partial E}{\partial \psi_M}
$$

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Again, there is an efficiency substitution effect and a benefit effect.

When ϕ_ℓ is considered instead of ψ_m, the procedure is the same. The results are somewhat different in that ϕ_ℓ and ψ_m appear differently in equations (1) through (7). The results are as follows with δ^* being the ϕ_ℓ version of δ.

$$
\begin{bmatrix} \dfrac{\partial E}{\partial \phi_\ell} \\[2ex] \delta^* \end{bmatrix} = \begin{bmatrix} (A_{11} - QR^{-1}P)^{-1} & -(A_{11} - QR^{-1}P)^{-1}QR^{-1} \\[2ex] -R^{-1}P(A_{11} - QR^{-1}P)^{-1} & R^{-1} + R^{-1}P(A - QR^{-1}P)^{-1}QR^{-1} \end{bmatrix} \begin{bmatrix} \xi^{Lx1} \\[2ex] 0^{I+J+K+3x1} \end{bmatrix}
$$

where ξ is a vector of zero's except for a minus one in the ℓth component.

Explicitly, the solution for δ^* is

$$
\delta^* = -R^{-1}P \frac{\partial E}{\partial \phi_\ell}
$$

and there is only a benefit effect.

The comparative static analysis performed in the preceding paragraphs yields some insight into a planning and programming system. However, much further research is needed to determine the qualitative properties of the systems of equations.

SECTION VII

EXPANSION-CONTRACTION PATHS

In most discussions of the firm found in textbooks on economic theory, the notion of an expansion path is discussed. This path is usually stated as a relationship among the inputs to the firm (e.g.,

see [13]) though some authors use a different characterization (e.g., see [14]). Whatever the chosen characterization, the path is derived by considering the minimum cost operating point for all levels of output. Such discussions of the expansion path do not consider the possible existence of physical assets at the start of the problem. These physical assets may be considered historical accidents for the decision problem of the firm. Also the discussions do not consider any production process that might be considered as a "stages of production" system. This section of the paper applies the expansion path notion to the planning and programming system model which requires consideration of assets and multiple "stages of production." In addition, the model in this paper contains multiple cost measures which must be considered.

The multiple cost measures are considered in the expansion path analysis by use of a matrix. As in the traditional economic theory of the firm, an expansion path is derived by considering multiple levels of a single cost measure. This is merely repeated for each measure yielding the expansion-contraction path matrix when all results are tabulated.

The expansion path when no physical assets are present at the beginning of the planning period will now be derived. The derivation will use the results of the efficiency problem discussed in Section IV. The derivation begins by applying the implicit function theorem to the necessary conditions (equation IV.1 - IV.7) of the efficiency problem. This results in the equations

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(1) $\underline{E} = \underline{E}(\underline{\psi}, \underline{\phi})$

(2) $\underline{x} = \underline{X}(\underline{\psi}, \underline{\phi})$

(3) $\underline{y} = \underline{Y}(\underline{\psi}, \underline{\phi})$

(4) $\underline{z} = \underline{Z}(\underline{\psi}, \underline{\phi})$

(5) $\underline{\lambda} = \underline{\Lambda}(\underline{\psi}, \underline{\phi})$.

In the case at hand, $\underline{\psi}$ is a vector of one component and since the
sum of the ϕ's and ψ's is one, there are only L independent
variables among the ψ's and ϕ's. But this permits the inversion
of (1) by another application of the implicit function theorem yielding

$$\underline{\phi} = \underline{\phi}(E)$$

Substituting these equations in (2) - (4), the only variables of
interest, yields the composite functions

$$\underline{x} = \underline{X}^*(E)$$

$$\underline{y} = \underline{Y}^*(E)$$

$$\underline{z} = \underline{Z}^*(E) .$$

These are a vector function characterization of the expansion path of
the planning and programming model. These vector functions are the rows
in the expansion-contraction path matrix.

As the reader has undoubtedly noted, the previous equations suppress
the cost model coefficients. If these are present, then it may be pos-
sible to reduce the vector function to a single function. This is not
done here because the exact structure of the cost model becomes of
importance in determining the number and arrangement of the coefficients.
Also, the coefficients are frequently random variables so the interpreta-
tion of the resulting single function must be considered carefully and
this would greatly extend the length of the present paper. Hence, the

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"complete collapsing" to a single equation characterization is not included in this paper.

The discussion now turns to the inclusion of existing physical assets, sometimes called free assets, into the analysis. This involves no great change in the results of the analysis since the variables in the model are all time-dated. Hence, by inclusion of a fixed upper bound on the quantities available of certain stock dimensioned and time zero dated variables, the model is extended to include "free assets." This change will result in the first order conditions containing additional inequalities for each of the "free assets." If some "free asset" is not demanded to the extent of its supply, then the preceding sections remain unchanged. If all the supply is used at an efficient point, then for the development of the cost-benefit function and the expansion paths, variables should be treated as a first-order exogenous variable [15]. The results then have the same form but with some variables interpreted as first-order exogenous variables. When discussing parameter variations, there are no basic difficulties though care must be exercised since some derivatives may only be left or right (but not both) derivatives. Also, the decision rules associated with (4) in Section IV change since for those free assets with a positive internal opportunity cost (IV.4) becomes

$$IV_m C_{mz_{k^*}} + \lambda_2 G_{z_{k^*}} - \lambda_{k^*} \leq 0$$

$$z_{k^*}(IV_m C_{mz_{k^*}} + \lambda_2 G_{z_{k^*}} - \lambda_{k^*}) = 0 \qquad z_{k^*} \geq 0$$

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where λ_{k^*} is the internal opportunity cost of the k^{*th} free asset. Hence, the total marginal cost as computed in the cost sub-model must be increased by the internal opportunity of the "free assets" to obtain the overall marginal cost. It is noted that in this interpretation the costs are treated as negative numbers and basic resources as positive numbers.

The above discussion shows that the "free assets" that may exist at the beginning of the planning horizon do not cause a major revision of the results. The complications are rather easily accommodated. However, it is now useful to consider the language of expansion and contraction paths to denote the relative direction from the current position.

SECTION VIII

SUMMARY

In this paper, a model of a planning and programming system has been formulated and exercised. The model includes a superordinate goal of efficiency imposed by the departmental management. This superordinate goal and the basic structural model of a planning and programming system are used to study the decision rules needed for efficiency. These decision rules are interpreted by use of a numéraire, a unit of account and efficiency prices. When so-called "free assets" are present, the decision rules are given an additional interpretation by use of the "internal" opportunity cost of the "free" asset. While no imperative sentences are constructed from the declarative sentences associated with the decision rules, such could be done. On the logical problems associated with this translation, see Herbert A. Simon [16].

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The comparative statics technique (sensitivity analysis) when applied to the model results, in general, in the derivation of a benefit effect and a substitution effect when a parameter varies. These benefit and substitute effects are the basis for the definition of efficiency substitutes and compliments. These effects in a nonbudget constrained model may be somewhat surprising to the reader.

The information that is relevant for the top management of the department is characterized in the form of an overall cost-benefit function and an expansion contraction path matrix. These characterizations of the information are based on the use of the implicit function theory and hence are local results.

Many areas of future research effort are feasible. The reader will undoubtedly have derived a list for himself. The model presented in this paper could be extended by improving the technological relationships in the submodels by explicitly considering risk, and by seeking more global results for the information summary functions. Also, the use of myopic (e.g., one period) decision rules and the effect of such on the system could be investigated. Finally, the use of a superordinate goal may be replaced with a model of the conflict resolution process internal to the planning and programming system. Some of these areas are currently under study by the author.

As must be clear to the reader, this is an exploratory paper in an area where much research remains to be done. Currently, the government institutionalizes various mechanisms of the planning process with little guidance available on the systems performance. Also, given that planning and programming systems are in expanding use, the effect on

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the economy as a whole is of interest. However, all these broader
analyses await some more basic research on the systems themselves.
The author hopes that he has enticed some readers to consider this
research area.

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13. ABSTRACT

A complete theory of cost-benefit analysis would include a model of a governmental department's "optimum program package" as revealed by a planning, programming and budgeting (PPB) system. The literature on such a model is limited. In this paper a model of the planning and programming aspects of a PPB system is formulated and investigated.

The model incorporates the component parts of the PPB system structure and the multiplicity of benefit and cost measures, both commensurable and incommensurable, so associated. The variables have attributes for time, space, state-of-nature, and physical-social characteristics. A vector nonlinear programming formulation, together with implicit function theory, yields decision rules characterizing the choice of an efficient program package. These rules reflect the usefulness of a program package numeraire and indicate that many of the rules usually discussed are at least rather misleading.

Application of the usual comparative statics technique yields an equation set relating endogenous variable variations to parameter variations. For many parameters, e.g., cost estimating coefficients, efficiency substitution and benefit effects can be derived and used to define efficiency substitutes and complements.
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13. ABSTRACT

The necessary conditions of the programming problem and implicit function theory are used to derive a program package cost-benefit function. This function relates benefit and cost measures and summarizes all planning and programming alternatives. For a given program package decision, the expansion-contraction paths for the department are also derived and characterized. There are multiple paths due to the use of multiple measures of cost. The influence on the expansion-contraction paths of the existence of assets at the beginning of the planning period is studied.

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